15 Laplace transform. Basic properties

We spent a lot of time learning how to solve linear nonhomogeneous ODE with constant coefficients. However, in all the examples we have considered, the right hand side (function $f(t)$) was continuous. This is not usually so in the real world applications. In particular, we can consider the differential operator L as a black box, which receives as input the external signal $f(t)$ and produces as output the solution $y(t)$, symbolically, $y = L^{-1}f$. It is more often than not the external signal can be represented as a *piecewise continuous function*, hence it would be of great value to have an efficient method to solve such problems. In the next three lectures we will learn one such possible method, which is based on the Laplace transform.

Definition 1. *Let function f be defined on $[0,\infty)$. Then its Laplace transform $\mathscr{L}\{f\}$ is another function F, which is defined as*

$$F(s) = \mathscr{L}\{f\} := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t. \tag{1}$$

The Laplace transform, according to this definition, is an operator: It is defined on functions, and it maps functions to another functions. Generally s is a complex variable, but in most of the examples we consider, we will not bother about the domain of F or about the question on the existence of the Laplace transform, for all the functions we deal with their Laplace transforms are well defined.

Note that in (1) the Laplace transform is defined as an *improper integral*. Strictly speaking, while evaluating this integral, we need to consider the limit

$$\lim_{c\to\infty} \int_0^c e^{-st} f(t)\,\mathrm{d}t.$$

It is a good idea to remember about this limit, but in the calculations that follow I will usually use shortcut notations. Let me start with several examples.

Example 2. Let $f(t) = 1$. Find $\mathscr{L}\{f\}$. By definition,

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\,\mathrm{d}t = -\frac{1}{s}e^{-st}|_0^\infty = \frac{1}{s}\,.$$

Example 3. Let $f(t) = e^{at}$ for some $a \in \mathbf{R}$. Using the integration,

$$\mathscr{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = -\frac{1}{s-a}e^{-(s-a)t}|_0^\infty = \frac{1}{s-a}\,,$$

where I assumed that $s > a$.

Example 4. Let $f(t) = t$. Here I will use integration by parts:

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}t\,\mathrm{d}t = -\frac{t}{s}e^{-st}|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s^2}\,.$$

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

Example 5. Find $\mathscr{L}\{\sin t\}$ and $\mathscr{L}\{\cos t\}$. Probably you remember that to evaluate the integrals of the form $\int e^{at}\sin bt\,\mathrm{d}t$ or $\int e^{at}\cos bt\,\mathrm{d}t$ you need to use a clever trick with integration by parts twice (do it). I, however, will choose a different approach. From Euler's formula $e^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$, we can find that

$$\sin t = \frac{e^{\mathrm{i}t} - e^{-\mathrm{i}t}}{2\mathrm{i}}, \quad \cos t = \frac{e^{\mathrm{i}t} + e^{-\mathrm{i}t}}{2}.$$

Now,

$$\mathscr{L}\{\sin t\} = \frac{1}{2\mathrm{i}}\int_0^\infty (e^{\mathrm{i}t} - e^{-\mathrm{i}t})e^{-st}\,\mathrm{d}t = \frac{1}{2\mathrm{i}}\left(\frac{1}{s-\mathrm{i}} - \frac{1}{s+\mathrm{i}}\right) = \frac{1}{s^2+1}.$$

Analogously,

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}.$$

We can continue evaluating these integrals and extending the list of available Laplace transforms. However, a much more powerful approach is to infer some general properties of the Laplace transform, and use them, instead of calculating the integrals. First very useful property is the linearity of the Laplace transform:

1° *Linearity.* $\mathscr{L}$ is a linear operator. This means that for any two functions f and g for which the Laplace transform is defined, and two constants $a, b \in \mathbf{R}$ we have

$$\mathscr{L}\{af + bg\} = a\,\mathscr{L}\{f\} + b\,\mathscr{L}\{g\}.$$

This follows from the linearity of the integrals.

Example 6. Using this property we can easily find, using the information above, the Laplace transform of, e.g., $5 - 3t + \pi\cos t$:

$$\mathscr{L}\{5 - 3t + \pi\cos t\} = 5\,\mathscr{L}\{1\} - 3\,\mathscr{L}\{t\} + \pi\,\mathscr{L}\{\cos t\} = \frac{5}{s} - \frac{3}{s^2} + \frac{\pi s}{s^2+1}.$$

2° *Shifting property.* If $\mathscr{L}\{f\} = F(s)$ then $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$.

To prove this property, consider

$$\begin{aligned}\mathscr{L}\left\{e^{at}f(t)\right\} &= \int_0^\infty e^{at}f(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t \qquad (\text{let } p = s-a) \\ &= \int_0^\infty e^{-pt}f(t)\,\mathrm{d}t = F(p) = F(s-a).\end{aligned}$$

Example 7. Now, to find, e.g., $\mathscr{L}\left\{e^{3t}\sin t\right\}$ we do not need to evaluate the integral:

$$\mathscr{L}\left\{e^{3t}\sin t\right\} = \frac{1}{(s-3)^2+1},$$

since $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$.

3° *Time scaling.* Let $\mathscr{L}\{f(t)\} = F(s)$ then

$$\mathscr{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right),$$

for any $a > 0$.

To prove, we start again with the definition

$$\mathscr{L}\{f(at)\} = \int_0^\infty f(at)e^{-st}\,\mathrm{d}t,$$

and use the change of variables $at = \tau$, from where $\mathrm{d}t = \mathrm{d}\tau/a$. Note that for $a > 0$ the limits of integration will not change:

$$\mathscr{L}\{f(\tau)\} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}\,\mathrm{d}\tau = \frac{1}{a}F\left(\frac{s}{a}\right)$$

as required. Note that if we allow any sign for $a \neq 0$ then (prove)

$$\mathscr{L}\{f(at)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right).$$

Example 8. Find $\mathscr{L}\{\cos 3t\}$. By the previous property and the fact that $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$ we find

$$\mathscr{L}\{\cos 3t\} = \frac{1}{3}\frac{s/3}{(s/3)^2+1} = \frac{s}{s^2+3^2}.$$

4° *Differentiation of the frequency.* Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\{tf(t)\} = -F'(s).$$

Or, more generally,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad n \in \mathbf{N}.$$

To show that this is true, consider the derivative of $F(s)$:

$$F'(s) = \int_0^\infty (-t)f(t)e^{-st}\,\mathrm{d}t,$$

which implies the property.

Example 9. What is $\mathscr{L}\left\{t^3\right\}$? We can evaluate the integral, but it is easier to find, using Property 4 and the fact that $\mathscr{L}\{1\} = 1/s$, that

$$\mathscr{L}\left\{t^3\right\} = (-1)^3\left(\frac{1}{s}\right)''' = \frac{3\cdot 2}{s^4}.$$

5° *Differentiation.* Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0).$$

For the proof, consider

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}\,\mathrm{d}t$$

and use the integration by parts

$$\mathscr{L}\left\{f'(t)\right\} = e^{-st}f(t)|_0^\infty + s\int_0^\infty f(t)e^{-st}\,\mathrm{d}t = -f(0) + s\,\mathscr{L}\{f\} = sF(s) - f(0).$$

We can generalize this property to differentiation of any order. For instance, to find $\mathscr{L}\{f''(t)\}$ just consider $f''(t)$ as the derivative of $f'(t)$ for which we already found the Laplace transform $sF(s) - f(0)$. Hence, according to the property,

$$\mathscr{L}\left\{f''(t)\right\} = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0).$$

It is useful to make a separate table with properties and Laplace transforms of frequently occurring functions.

Inverse Laplace transform. If we are given a function f we can find its Laplace transform by evaluating the corresponding integral:

$$F(s) = \mathscr{L}\{f(t)\}.$$

It is also possible to go in the opposite direction: We are given $F(s)$ and asked to find a function $f(t)$, for which $f = \mathscr{L}^{-1}\{F\}$, i.e., find the *inverse Laplace transform.* This is possible due to the following important uniqueness theorem

Theorem 10. *If two functions f_1 and f_2 have the same Laplace transform, then they coincide at every point t at which they both are continuous.*

There exists a general formula for finding the inverse Laplace transform:

$$\mathscr{L}^{-1}\{F\} = \frac{1}{2\pi\mathrm{i}}\lim_{T\to\infty}\int_{\gamma-\mathrm{i}T}^{\gamma+\mathrm{i}T} e^{st}F(s)\,\mathrm{d}s,$$

but we will never use it (if you'd like to understand what this formula means, consider taking a course on Complex Analysis).

Our algorithm of finding the inverse Laplace transform is by using the table. As an example, we know that

$$\mathscr{L}\{\cos 4t\} = \frac{s}{s^2+16}.$$

This means that

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t.$$

While finding the inverse Laplace transform it is important to remember that it is also linear. Therefore,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+16}\right\} = \mathscr{L}^{-1}\left\{\frac{4\cdot\frac{1}{4}}{s^2+4^2}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \frac{1}{4}\sin 4t.$$

More about it in the next lecture.

15.1 Table of the most useful Laplace transforms

In this section I list the Laplace transforms of the most frequently encountered functions.

$f(t),\ t \in [0, \infty)$	$F(s) = \mathscr{L}\{f\}$
1	$\dfrac{1}{s}$
$e^{at},\ a \in \mathbf{R}$	$\dfrac{1}{s-a}$
$t^n,\ n \in \mathbf{N}$	$\dfrac{n!}{s^{n+1}}$
$\sin \omega t,\ \omega \in \mathbf{R}$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t,\ \omega \in \mathbf{R}$	$\dfrac{s}{s^2+\omega^2}$
$\sinh \omega t,\ \omega \in \mathbf{R}$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t,\ \omega \in \mathbf{R}$	$\dfrac{s}{s^2-\omega^2}$
$e^{at} \sin \omega t,\ a, \omega \in \mathbf{R}$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
$e^{at} \cos \omega t,\ a, \omega \in \mathbf{R}$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sinh \omega t,\ a, \omega \in \mathbf{R}$	$\dfrac{\omega}{(s-a)^2-\omega^2}$
$e^{at} \cosh \omega t,\ a, \omega \in \mathbf{R}$	$\dfrac{s-a}{(s-a)^2-\omega^2}$
$t \sin \omega t,\ \omega \in \mathbf{R}$	$\dfrac{2\omega s}{\left(s^2+\omega^2\right)^2}$
$t \cos \omega t,\ \omega \in \mathbf{R}$	$\dfrac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$
$t^n e^{at},\ a \in \mathbf{R}, n \in \mathbf{N}$	$\dfrac{n!}{(s-a)^{n+1}}$
$u(t)$, Heaviside function	$\dfrac{1}{s}$